Multisoliton Excitations for the Kadomtsev-Petviashvili Equation

Zheng-Yi Ma[a,b], Guo-Sheng Hua[a], and Chun-Long Zheng[a,b]

[a] College of Science, Zhejiang Lishui University, Zhejiang Lishui 323000, P. R. China
[b] Shanghai Institute of Applied Mathematics and Mechanics,
Shanghai University, Shanghai 200072, P. R. China

Reprint requests to Prof. Z.-Y. M.; E-mail: mazhengyi_77@yahoo.com.cn

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By means of the standard truncated Painlevé expansion and a special Bäcklund transformation, some exact multisoliton solutions are derived for the Kadomtsev-Petviashvili equation. The evolution properties of the multisoliton excitations are investigated and some novel features or interesting behaviors are revealed. The results show that four straight-line solitons are annihilated or produced with the time increases, which is very similar to the completely nonelastic collision among electrons and positrons.

Key words: Kadomtsev-Petviashvili Equation; Variable Separation Approach; Bäcklund Transformation; Annihilation.

1. Introduction

Lately, owing to a seed function, the multilinear variable separation approach (MLVSA) has been developed well for many (2+1)-dimensional integrable systems [1] such as the Davey-Stewartson (DS) equation [2], the Nizhnik-Novikov-Veselov(NNV) equation [3], the asymmetric NNV (ANNV) equation [4], the asymmetric DS (ADS) equation [5], the dispersive long wave equation (DLWE) [6], the Broer-Kaup-Kupershmidt (BKK) system [7], the higher-order BKK system [8], the nonintegrable (2+1)-dimensional Korteweg-deVries (KdV) equation [9], the long wave-short wave interaction model (LWSWIM) [10], the Maccari system [11], the Burgers equation [12], the (2+1)-dimensional sine-Gordon (2DsG) system [13] and the general (N+M)-component AKNS system [14]. Using the MLVSA results, scientists have found that the formula

$$\mathbb{U} = \frac{2(a_0a_3 - a_1a_2)p_xq_y}{(a_0 + a_1p + a_2q + a_3pq)^2} \tag{1}$$

is valid for suitable fields or potentials of all the above-mentioned models. In (1), $p \equiv p(x,t)$ is an arbitrary function of $\{x,t\}$; $q \equiv q(y,t)$ may be either an arbitrary function for some kinds of models such as the DS system, the NNV system and the 2DsG system or an arbitrary solution of a Riccati equation (or heat conduction equation) for some other systems; a_0, a_1, a_2 and a_3 are constants.

Because of the arbitrariness of the functions p and q included in the universal formula (1), various types of multiple localized excitations with or without completely interacting behavior can be constructed. Two types of the completely nonelastic behaviors – the soliton fusion and fission phenomena – were well studied [7 – 9,15,16]. That is to say, for a physical model, two or more solitons may fuse into one soliton at a special time, while sometimes one soliton may fission into two or more solitons at other special time. To our knowledge, however, another nonelastic behavior – the soliton annihilable phenomenon – has seldom been mentioned up to now. In this paper, we concentrate our attention on the MLVSA for the Kadomtsev-Petviashvili (KP) equation. Although we couldn't find the suitable seed function, some special types of multisoliton solutions of the KP equation are derived via a Painlevé-Bäcklund transformation and the MLVSA. Except for several existing classical solitons, the results show that four straight-line solitons are annihilated or produced with the time increasing, which is very similar to the completely nonelastic collision among electrons and positrons.

It is shown in Section 2 that the novel localized solutions for the KP equation are first derived; and then, several classical solitons and the soliton annihilable and producing phenomena of the KP equation are revealed. A short summary and discussion is presented in Section 3.

2. Novel Soliton Solution of the KP Equation and its Localized Structures

The KP equation

$$u_t = u_{xxx} + 6uu_x + 3\partial_x^{-1} u_{yy} \tag{2}$$

was first introduced by Kadomtsev and Petviashvili [17] in order to study the stability of one-dimensional solitons against transverse perturbation. The multisoliton solutions for the KP equation were obtained by various methods, for instance, the inverse scattering method [18], Hirota method [19], bilinear Bäcklund transformation [20], trace method [21], Wronskian technique [22].

Here, for simplicity, we introduce a transformation $v = \partial_x^{-1} u_y$ and change the KP equation into a set of two coupled nonlinear partial differential equations:

$$u_t = u_{xxx} + 6uu_x + 3v_y, \tag{3}$$

$$u_y = v_x. \tag{4}$$

We consider the following Painlevé-Bäcklund transformation for u and v in (3) and (4):

$$u = 2(\ln f)_{xx}, \quad v = 2(\ln f)_{xy}, \tag{5}$$

which can be derived from the standard truncated Painlevé expansion [$f = f(x,y,t)$ is a function of variable x, y and t to be determined]. Substituting (5) into (3) yields a trilinear equation of f, while (4) is satisfied identically under the above transformation, which reads

$$\begin{aligned}&(f_{xxt} - f_{xxxxx} - 3f_{xyy})f^2 + (5f_{xxxx}f_x - 2f_{xxx}f_{xx}\\ &- f_{xx}f_t + 6f_{xy}f_y - 2f_xf_{xt} + 3f_xf_{yy})f + 6f_{xx}^2 f_x\\ &-8f_{xxx}f_x^2 + 2f_x^2 f_t - 6f_xf_y^2 = 0.\end{aligned} \tag{6}$$

We take f as

$$\begin{aligned}f(x,y,t) = a_0 + a_1 p(x,t) + a_2 q(y,t)\\ + a_3 p(x,t) q(y,t),\end{aligned} \tag{7}$$

where the variable separation functions $p \equiv p(x,t)$ and $q \equiv q(y,t)$ are functions of the indicated variables, while a_0, a_1, a_2, a_3 are arbitrary constants.

Inserting the ansatz (7) into (6), we obtain two variable separated equations:

$$p_t = 4p_{xxx} + cp_{xx}, \tag{8}$$

$$q_t = -cq_y, \tag{9}$$

where c is an arbitrary constant.

Equation (9) has the general solution

$$q(y,t) = q(a(y-ct)+b), \tag{10}$$

where $q(a(y-ct)+b)$ is an arbitrary function of $a(y-ct)+b$ and a, b are two arbitrary constants.

As to (8), although we can't derive its general solution, when taking $\xi = kx + lt + m$ (the constants k, l, m will be determined later), we obtain:

a)

$$p = (C_1 + C_2\xi)\mathrm{e}^{-\frac{c}{8k}\xi} + C_3, \quad \text{if} \quad \Delta = k(kc^2 + 16l) = 0; \tag{11}$$

b)

$$p = C_1 \mathrm{e}^{\frac{-ck+\sqrt{k(kc^2+16l)}}{8k^2}\xi} + C_2 \mathrm{e}^{\frac{-ck-\sqrt{k(kc^2+16l)}}{8k^2}\xi} + C_3, \quad \text{if} \quad \Delta = k(kc^2+16l) > 0; \tag{12}$$

c)

$$p = \mathrm{e}^{-\frac{c}{8k}\xi}\left(C_1 \cos\frac{\sqrt{-k(kc^2+16l)}}{8k^2}\xi + C_2 \sin\frac{\sqrt{-k(kc^2+16l)}}{8k^2}\xi\right) + C_3, \quad \text{if} \quad \Delta = k(kc^2+16l) < 0. \tag{13}$$

The constants C_1, C_2, C_3 in (11)–(13) are arbitrary.

Finally, we obtain the MLVSA solution of the KP equation as follows:

$$u = \frac{2(a_1 + a_3q)((a_0 + a_1p + a_2q + a_3pq)p_{xx} - (a_1 + a_3q)p_x^2)}{(a_0 + a_1p + a_2q + a_3pq)^2}, \tag{14}$$

$$v = \frac{2(a_0a_3 - a_1a_2)p_xq_y}{(a_0 + a_1p + a_2q + a_3pq)^2}, \tag{15}$$

where the related functions $p \equiv p(x,t)$, $q \equiv q(y,t)$ are expressed by (11)–(13) and (10), respectively.

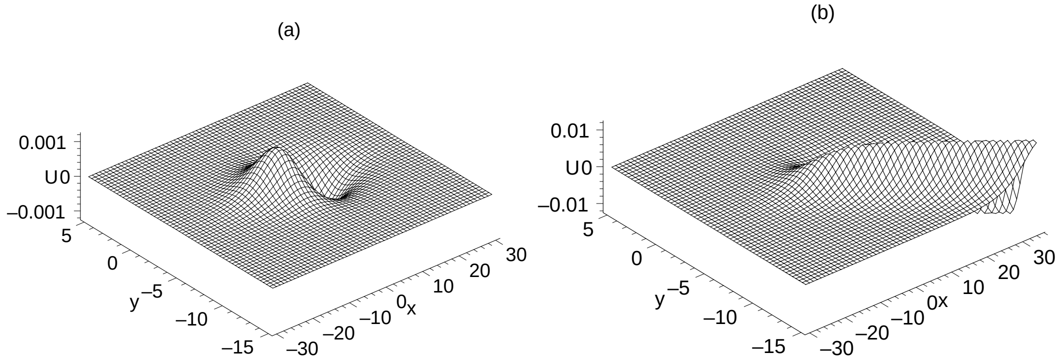

Fig. 1. (a) Plot of a pair of dromions for the physical quantity U expressed by (17) at time $t = 0$; (b) Plot of a pair of solitoffs for the physical quantity U expressed by (18) at time $t = 0$.

In a (2+1)-dimensional system, one of the most important matters are the existing localized coherent structures, such as dromions, solitoffs and so on. Here, we present several typical localized excitations for the KP equation through the potential $U \equiv u_y$, where u is expressed by (14).

We know that a dromion excitation is localized in all directions. When taking the functions p in (11) and q in (10) as

$$p = (C_1 + C_2\xi)\mathrm{e}^{-\frac{c}{8k}\xi} + C_3, q = \mathrm{e}^{K(y-ct)+M}, \tag{16}$$

and the coefficients as $C_1 = 1, C_2 = C_3 = 0$, $a_0 = 1$, $a_1 = 2$, $a_2 = 3$, $a_3 = 4$, $k = -2$, $l = 0.5$, $m = 0$, $K = 1$, $M = 4$ and $c = 2$, we get

$$U = -\frac{\mathrm{e}^{-\frac{1}{4}x+y-\frac{31}{16}t+4} - 2\mathrm{e}^{-\frac{1}{2}x+y-\frac{15}{8}t+4} + 3\mathrm{e}^{-\frac{1}{4}x+2y-\frac{315}{8}t+8} - 4\mathrm{e}^{-\frac{1}{2}x+2y-\frac{31}{8}t+8}}{4\left(1 + 2\mathrm{e}^{-\frac{1}{2}x+\frac{1}{16}t} + 3\mathrm{e}^{y-2t+4} + 4\mathrm{e}^{-\frac{1}{4}x+y-\frac{31}{16}t+4}\right)^3}, \tag{17}$$

which is localized in all directions to the fixed time t. Figure 1a shows the structure of a pair of dromions for the quantity U at the time $t = 0$.

A single solitoff, which is called a half-straight-line soliton solution, can be depicted when the functions p and q are just as (16) and the coefficients are $C_1 = 1, C_2 = C_3 = 0$, $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $k = -2, l = 0.5$, $m = 0$, $K = 1$, $M = 4$ and $c = 2$. Figure 1b shows the structure of a pair of solitoffs for the physical quantity U expressed by

$$U = \frac{\mathrm{e}^{-\frac{1}{2}x+y-\frac{15}{8}t+4} - 2\mathrm{e}^{-\frac{1}{4}x+2y-\frac{63}{16}t+8} + 3\mathrm{e}^{-\frac{1}{2}x+2y-\frac{31}{8}t+8}}{4\left(\mathrm{e}^{-\frac{1}{4}x+\frac{1}{16}t} + 2\mathrm{e}^{y-2t+4} + 3\mathrm{e}^{-\frac{1}{4}x-y-\frac{31}{16}t+4}\right)^3}, \tag{18}$$

at the time $t = 0$.

When taking the functions p in (12) and q in (10) as

$$p = C_1\mathrm{e}^{\frac{-ck+\sqrt{k(kc^2+16l)}}{8k^2}\xi} + C_2\mathrm{e}^{\frac{-ck-\sqrt{k(kc^2+16l)}}{8k^2}\xi} + C_3, \quad q = \mathrm{e}^{\frac{1}{15}y^2-5} + \mathrm{e}^{\frac{1}{15}\sin y^2}, \tag{19}$$

and the coefficients as $C_1 = 1, C_2 = C_3 = 0$, $a_0 = a_1 = a_2 = 1$, $a_3 = 0$, $k = -0.08$, $l = -0.002048$, $m = 5$, and $c = 0$, we get a more interesting structure for the physical quantity U than we have ever seen before. That is, there exists an evolution from dromions to solitoffs. Figure 2a shows the whole structure for the physical quantity U at the time $t = 0.9$. To see more clearly, we divide the region $y \in [0, 18]$ into three suc-

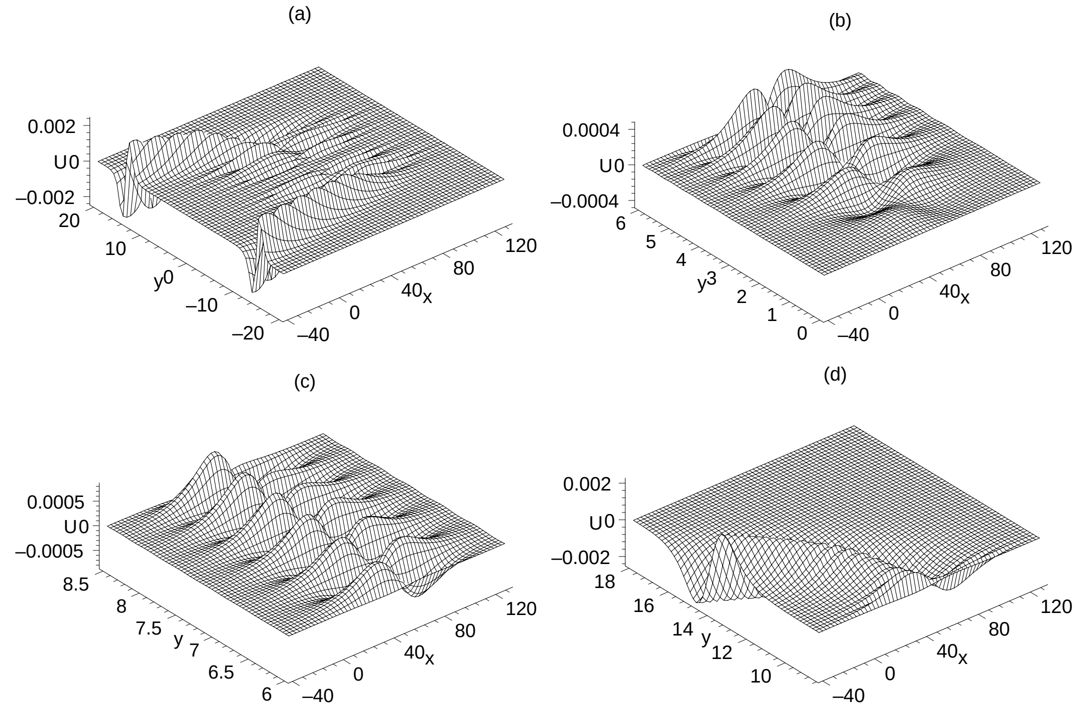

Fig. 2. (a) Plot of an evolution from dromions to solitoffs for the physical quantity U with the condition (19) at time $t = 0.9$. (b), (c), (d) Plots of the vivid structure in successive regions $y \in [0,6], [6,8.5]$ and $[8.5,18]$, respectively, corresponding to (a).

cessive regions $y \in [0,6], [6,8.5]$ and $[8.5,18]$, and we can see the vivid evolution structure from Figs. 2a, b and d, respectively.

In fact, we can derive a general exponential solution for the functions p and q of (8) and (9), that is

$$p = c_0 + \sum_{i=1}^{M} c_i \mathrm{e}^{k_i x + k_i^2(4k_i + c)t + m_i}, \tag{20}$$

$$q = C_0 + \sum_{i=1}^{N} C_i \mathrm{e}^{K_i(y - ct) + M_i}, \tag{21}$$

where c_0, c_i, k_i, m_i, c, C_0, C_i, K_i and M_i are arbitrary constants and M, N are positive integers.

At present, we turn our attention to the annihilable behavior of the multisolitons for the KP equation. For example, if selecting the constants in (20) and (21) as

$$\begin{aligned} &M = N = 2, \\ &c_0 = c_1 = c_2 = k_1 = m_1 = m_2 = 1,\ k_2 = -1, \\ &C_0 = C_1 = C_2 = K_2 = M_1 = M_2 = 1,\ K_1 = -1, \end{aligned} \tag{22}$$

we can derive a annihilable local coherent structure for the physical quantity U depicted in Figure 3. Here, we set $a_0 = 10$, $a_1 = a_2 = -1$, $a_3 = 1$ and $c = 2$, so that the physical quantity

$$\begin{aligned} U = &-18\left(\mathrm{e}^{-y+2t+1} - \mathrm{e}^{y-2t+1}\right)\left(9\mathrm{e}^{-x-2t+1}\right. \\ &+ 9\mathrm{e}^{x+6t+1} - \mathrm{e}^{2x-y+14t+3} - \mathrm{e}^{2x+y+10t+3} \\ &- \mathrm{e}^{-2x-y-2t+3} - \mathrm{e}^{-2x+y-6t+3} + 6\mathrm{e}^{-y+6t+3} \\ &\left.+ 6\mathrm{e}^{y+2t+3}\right)/\left(9 + \mathrm{e}^{x-y+8t+2} + \mathrm{e}^{x+y+4t+2}\right. \\ &\left.+ \mathrm{e}^{-x-y+2} + \mathrm{e}^{-x+y-4t+2}\right)^3. \end{aligned} \tag{23}$$

From Fig. 3, we can see that the four straight-line solitons are annihilated with the time increasing. This novel phenomenon indicates that their interaction is completely nonelastic, which is very similar to the completely nonelastic collision among electrons and positrons.

Along with this line and using the similar analysis, when taking

$$\begin{aligned} &M = N = 2, \\ &c_0 = c_1 = c_2 = k_1 = m_1 = m_2 = 1,\ k_2 = -1, \\ &C_0 = C_1 = C_2 = K_1 = M_1 = M_2 = 1,\ K_2 = -1, \\ &a_0 = 4.5, \quad a_1 = a_3 = 1, \quad a_2 = -1, \quad c = -2, \end{aligned} \tag{24}$$

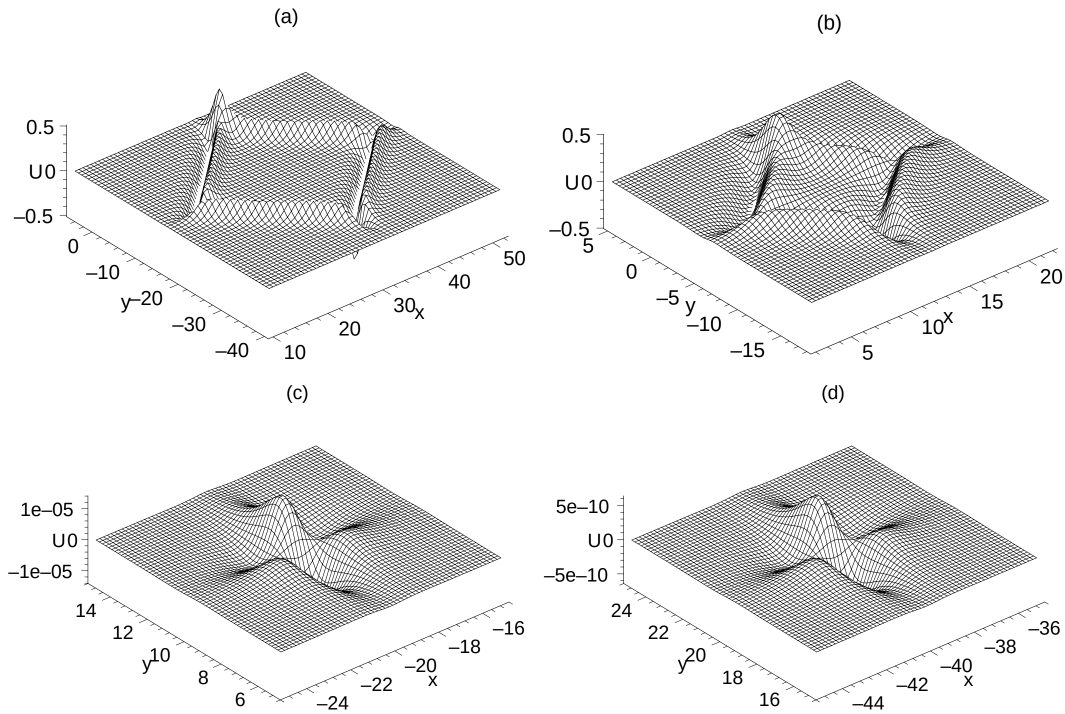

Fig. 3. The annihilable evolution of four straight-line solitons for the physical quantity U (23) at times: (a) $t = -8$; (b) $t = -3$; (c) $t = 5$; (d) $t = 10$.

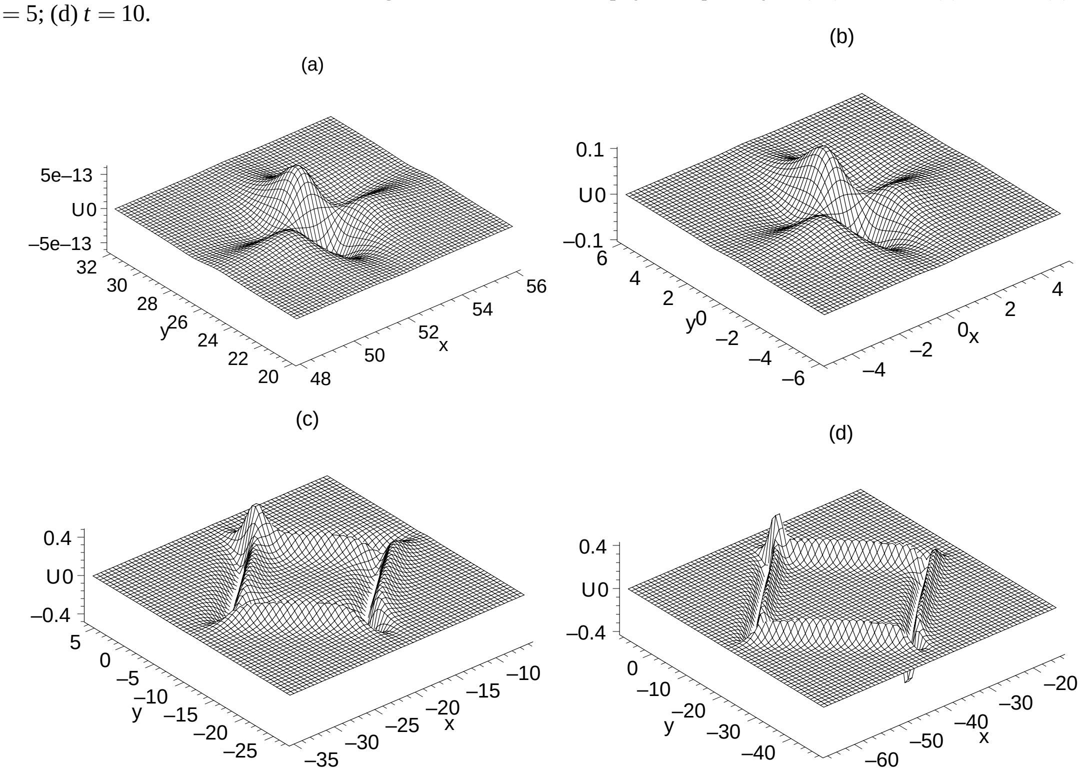

Fig. 4. The producing evolution of four straight-line solitons for the physical quantity U with the condition (24) at times: (a) $t = -13$; (b) $t = 0$; (c) $t = 5$; (d) $t = 10$.

one can find that four straight-line solitons for the physical quantity U are produced with the time increases. The corresponding plots are depicted in Figure 4.

3. Summary and Discussion

With the help of the Painlevé-Bäcklund transformation and the multilinear variable separation approach, some exact multisoliton solutions are derived for the KP equation. Based on the multisoliton excitations, several localized coherent soliton structures can be constructed by selecting appropriate functions. The evolution property among some local structures for the KP equation is discussed and the result show novel property and interesting behavior: the interaction among four straight-line solitons is annihilable or producing at different circumstances. This means that the interaction can be completely nonelastic, even though there was completely elastic behavior in some previous papers.

Although we have given some exact multisoliton solutions, it is obvious that there are still many significant and important problems waiting for further investigation. Such as: Can we prove what is a sufficient and necessary condition for the soliton annihilation/production? What is the general equation for the distribution of the energy and momentum after the soliton annihilation/production? Why does the evolution property of multisoliton excitations present different ways? How can one use the soliton annihilation/production of integrable models to study the practically observed soliton annihilation/production in the experiments? These are all the pending problems.

Thanks to the wide application of the soliton theory, to learn more about the localized excitations and their applications in reality it is worth to be studied further.

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[1] X. Y. Tang, S. Y. Lou, and Y. Zhang, Phys. Rev. E **66**, 046601 (2002).
[2] S. Y. Lou and J. Z. Lu, J. Phys. A: Math. Gen. **29**, 4209 (1996); S. Y. Lou, Phys. Scr. **65**, 7 (2002).
[3] S. Y. Lou, Phys. Lett. A **227**, 94 (2000).
[4] S. Y. Lou and H. Y. Ruan, J. Phys. A: Math. Gen. **34**, 237 (2001).
[5] H. Y. Ruan and Y. X. Chen, Acta Phys. Sin. **4**, 586 (2001) (in Chinese).
[6] M. Boiti, J. J. P. Leon, and F. Pempinelli, Inv. Probl. **3**, 371 (1987); C. L. Zheng, J. M. Zhu, J. F. Zhang, and L. Q. Chen, Commun. Theor. Phys. (Beijing, China) **39**, 261 (2003); J. F. Zhang, Z. M. Lu, and Y. L. Liu, Z. Naturforsch **58a**, 280 (2003).
[7] S. Y. Lou, J. Phys. A: Math. Gen. **35**, 10619 (2002); C. L. Zheng and L. Q. Chen, Chaos, Solitons & Fractals **24**, 1347 (2005).
[8] J. Lin and H. M. Li, Z. Naturforsch **57a**, 929 (2002); Z. Y. Ma, J. M. Zhu, and C. L. Zheng, Chin. Phys. **13**, 1382 (2004); Z. Y. Ma and C. L. Zheng, Commun. Theor. Phys. (Beijing, China) **43**, 993 (2005).
[9] X. Y. Tang and S. Y. Lou, Commun. Theor. Phys. (Beijing, China) **38**, 1 (2002); J. Lin and F. M. Wu, Chaos, Solitons & Fractals **19**, 189 (2004).
[10] M. Oikawa, M. Okamura, and M. Funakoshi, J. Phys. Soc. Jpn. **58**, 4416 (1989); M. Boiti, J. J. P. Leon, L. Matina, and F. Pempinelli, Phys. Lett. A **132**, 432 (1988); W. H. Huang, J. F. Zhang, and Z. M. Sheng, Chin. Phys. **11**, 1101 (2002); J. P. Fang, C. L. Zheng, and L. Q. Chen, Commun. Theor. Phys. (Beijing, China) **43**, 245 (2005).
[11] A. Maccari, J. Phys. A: Math. Gen. **38**, 4151 (1997); A. Uthayakumar, K. Nakkeran, and K. Porsezia, Chaos, Solitons & Fractals **10**, 1513 (1999); J. F. Zhang, Commun. Theor. Phys. (Beijing, China) **35**, 267 (2000).
[12] J. P. Ying and S. Y. Lou, Chin. Phys. Lett. **20**, 1448 (2003); C. L. Zheng, Commun. Theor. Phys. (Beijing, China) **41**, 391 (2004).
[13] S. Y. Lou, J. Phys. A: Math. Gen. **35**, 10619 (2002).
[14] S. Y. Lou, C. L. Chen, and X. Y. Tang, J. Math. Phys. **43**, 4078 (2002).
[15] C. L. Zheng, J. P. Fang, and L. Q. Chen, Commun. Theor. Phys. (Beijing, China) **41**, 903 (2004).
[16] S. Wang, X. Y. Tang, and S. Y. Lou, Chaos, Solitons & Fractals **21**, 231 (2004).
[17] B. B. Kadomtsev and V. I. Petviashvili, Sov. Phys. Dokl. **15**, 539 (1970).
[18] V. E. Zakharov and A. B. Shabat, Funct. Anal. Appl. **8**, 226 (1974); V. E. Zakharov and A. B. Shabat, Sov. Sci. Rev. A **1**, 133 (1980).
[19] J. Satsuma, J. Phys. Soc. Jpn. **40**, 286 (1976).

[20] R. Hirota, Direct Methods in Soliton Theory, Springer, Berlin 1980; S. F. Deng, Chaos, Solitons & Fractals **25**, 475 (2005).

[21] M. Wadati and K. Sawada, J. Phys. Soc. Jpn. **48**, 312 (1980); **48**, 319 (1980); K. Ohkuma and M. Wadati, J. Phys. Soc. Jpn. **52**, 749 (1983).

[22] N. C. Freeman and J. J. C. Nimmo, Phys. Lett. A **95**, 1 (1983).